

Imperial/TP/98-99/67
hep-th/9908105

Born-Infeld action, supersymmetry and string theory

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Abstract

We review and elaborate on some aspects of Born-Infeld action and its supersymmetric generalizations in connection with string theory. Contents: BI action from string theory; some properties of bosonic $D = 4$ BI action; $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric BI actions with manifest linear $D = 4$ supersymmetry; four-derivative terms in $\mathcal{N} = 4$ supersymmetric BI action; BI actions with ‘deformed’ supersymmetry from D-brane actions; non-abelian generalization of BI action; derivative corrections to BI action in open superstring theory.

To appear in the Yuri Golfand memorial volume, ed. M. Shifman, World Scientific (2000)

August 1999

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1. Introduction

It is a pleasure for me to contribute to Yuri Golfand's memorial volume. I met Yuri several times during his occasional visits of Lebedev Institute in the 80's. Two of our discussions in 1985 I remember quite vividly.

Golfand found appealing the interpretation of string theory as a theory of 'quantized coordinates', viewing it as a generalization of some old ideas of noncommuting coordinates. In what should be an early spring of 1985 he read our JETP Letter [1] which was a brief Russian version of our approach with Fradkin [2] to string theory effective action based on representation of generating functional for string amplitudes as Polyakov string path integral with a covariant 2-d sigma model in the exponent. In [1] our approach was interpreted in a somewhat heuristic way: (i) the basic quantized 'pre-field' is a set of coordinates x^m of a D -dimensional space which may be taken as a string (or membrane) coordinates depending on internal parameters σ^i (world-volume coordinates); (ii) the classical space-time coordinates are expectation or 'center-of-mass' values of $x^m(\sigma)$; (iii) all space-time fields are 'excitations' of these quantized coordinates, appearing as generalized 'sources' or 'couplings' in the path integral action, $\int d^2\sigma \sum \partial^{k_1} x^{m_1} \dots \partial^{k_n} x^{m_n} B_{m_1 \dots m_n}(x(\sigma))$. Golfand asked me if superstring theory should then be interpreted as a theory of quantized supercoordinates (x, θ) . I told him of a recent paper by Green and Schwarz [3] as the one that should provide a basis for such a program. Later in spring 1985 we generalized the sigma model approach to Green-Schwarz superstring [4] using its light-cone gauge formulation [5].

In summer of 1985 Golfand approached me again after having seen the Lebedev Institute preprint version of our paper [6] on the derivation of the Born-Infeld action from the open string theory. This was a simple application of the non-perturbative in number of fields approach of [2], allowing one for the first time to sum certain terms in the string effective action to all orders in α' . Yuri stressed the importance of the fact that the string tension $T = (2\pi\alpha')^{-1}$ is determining the critical value of the electric field. He was also excited about a possibility (noted in [6]) of a kind of 'bootstrap' if such a non-linear action following itself from string theory admits string-like solutions. In [6] we mentioned that vortex solutions of similar Born-Infeld type actions (e.g. $\sqrt{F^2}$) were discussed previously in [7]. This idea has similarity to some recent developments: a simple plane wave type solution of BI action ($F_{a0} + F_{a1} = 0$) may be re-interpreted ($A_a = TX_a$, $F_{a1} = T\partial_a X_1$) as a

fundamental string ending on D-brane [8,9] – the dimensional reduction of BI action along the 1-direction is simply the DBI action [10] describing D-brane collective coordinates.¹

I recall that Golfand was also asking me about supersymmetric extension of Born-Infeld action. At that time I was not aware of an early work [12] on this subject, but later in 1986 there appeared the paper [13] (inspired in part by the discovery of the relation of the bosonic Born-Infeld action to string theory) where $\mathcal{N} = 1$ supersymmetric version of $D = 4$ BI action was presented in the explicit form. While the requirement of $\mathcal{N} = 1$ $D = 4$ supersymmetry did not fix uniquely the bosonic part of the nonlinear abelian vector multiplet action [13] to be the standard $\sqrt{-\det(\eta_{mn} + T^{-1}F_{mn})}$, it is now clear that the condition of $\mathcal{N} = 4, D = 4$ (or $\mathcal{N} = 1, D = 10$) supersymmetry should imply this.²

It seems, therefore, that a review of some aspects of the Born-Infeld action and its supersymmetric extensions in the context of string theory is quite appropriate in this volume.

2. Born-Infeld action from string theory

The Born-Infeld action was derived from string theory in [6] as field strength derivative independent part of the open string effective action by starting with an ‘off-shell’ bosonic open string path integral on the disc in an external (abelian) vector field

$$Z(A) = \langle \text{tr } P \exp i \int d\varphi \dot{x}^m A_m(x) \rangle , \quad (2.1)$$

$$A_m(x_0 + \xi) \rightarrow \frac{1}{2}\xi^n F_{nm} + O(\partial F) ,$$

and using specific (ζ -function) renormalization to get rid of a linear divergence. In Appendix we repeat the original computation [6] of this partition function in the abelian $F_{mn} = \text{const}$ background in a slightly generalized form: we shall assume that the boundary part of the string action contains also the usual ‘particle’ term $M(\dot{x}^m)^2$. Here the

¹ This is of course not surprising from more stringy perspective, as T -duality along 1-direction should convert the wave momentum into the (wound) string charge. Since this solution is essentially a plane wave, it is also an exact solution of not only the BI action but also of the full open string theory effective action [11].

² In particular, the structure of F^4 [14,15] and, more non-trivially, F^6 [16] terms in the $\mathcal{N} = 1$ supersymmetric deformation of the $D = 10$ super Maxwell action was found to be exactly the same as in the BI action.

constant M may be viewed as an ‘off-shell’ condensate of a massive open string mode or simply as a formal regularization parameter. The effective boundary action will then have both a ‘first-derivative’ scale-invariant ($\sim T$) and a second-derivative ($\sim M$) parts and will interpolate between the string-theory case $T \neq 0$, $M = 0$ discussed in [6] and the standard particle case $T = 0$, $M \neq 0$ appearing in the Schwinger computation of $\log \det(-D^2(A))$. The resulting bosonic string partition function for a single magnetic field component F will be (superstring expression is similar, see Appendix)

$$Z \sim \frac{\Gamma(1 + \frac{T+iF}{M}) \Gamma(1 + \frac{T-iF}{M})}{[\Gamma(1 + \frac{T}{M})]^2}, \quad (2.2)$$

and thus will have the Born-Infeld $\sqrt{1 + (T^{-1}F)^2}$ and the Schwinger $\frac{\pi M^{-1}F}{\sinh(\pi M^{-1}F)}$ expressions as its $M = 0$ and $T = 0$ limiting cases.

The $F^2 + \alpha'^2 F^4$ terms in BI action were found to be in precise agreement with the ones derived directly from (super)string 4-point amplitude [17,18]. The reason why the renormalized open string path integral on the disc in $F_{mn} = \text{const}$ background reproduced, indeed, the correct effective action³ was explained in detail later [19,20]: the apparently missing Möbius group volume factor is only linearly divergent in the bosonic case (and is finite in the superstring case) and thus is effectively taken care of by the renormalization. The computation of the string partition function in a constant abelian background is essentially an ‘on-shell’ computation: $F_{mn} = \text{const}$ solves equations of motion for any gauge-invariant action $S(F)$ depending only on the field strength and its derivatives.⁴

Furthermore, in the important paper [21] it was demonstrated that the leading-order term in the expansion in ∂F of the condition of conformal invariance of the open string sigma model follows indeed from the BI action. In particular, $F_{mn} = \text{const}$ background defines a conformal 2-d field theory. The superstring generalization of this conformal invariance argument implied [22] that the derivative-independent term in the open superstring effective action should also be given by the same BI action. After some initial confusion in

³ It could seem that computing the partition function we were not dividing over Möbius group volume, compared to the standard on-shell generating functional for string S-matrix.

⁴ Related point is that the BI action is unambiguous: it is not changed by local field redefinitions of gauge potential since these lead to terms containing derivatives of F_{mn} which, by definition, are not included in the BI action. This is also related to the fact that the string partition function in $F_{mn} = \text{const}$ background does not contain logarithmic divergences.

[17] (corrected in errata) this conclusion was reached [16] also in the original path integral approach of [2].

Born-Infeld action is a unique example of the case when certain α' string corrections can be summed to all orders. Though no similar action is known in the closed-string context⁵ the expectation is that string tension defines a natural maximal scale for all field strengths, including curvature. The analogy with open string theory suggests that higher order α' terms in the effective action may eliminate (at least some) black hole singularities [24]. While in the Maxwell theory the field of a point-like charge is singular at the origin and its energy is infinite, in the Born-Infeld theory the electric field of a δ -function source is regular at $r = 0$ (where it takes its maximal value) and its total energy is finite [25]. From the point of view of the distribution of the electric field ($\rho_{\text{eff}} = \frac{1}{4\pi} \text{div } E$) the source is no longer point-like but has an effective radius $r_0 \sim \sqrt{\alpha'}$ (for example, in 4 dimensions $E_r = F_{rt} = \frac{Q}{\sqrt{r^4 + r_0^4}}$, $r_0^2 = 2\pi\alpha'Q$). Since both open and closed string theories are effectively non-local with characteristic scale of $\sqrt{\alpha'}$, one may expect that Schwarzschild singularity is smeared in a similar way.

The remarkable second advent of D-branes [26] four years ago [27] brought the Born-Infeld action again into the spot-light. In [10] the Born-Infeld action found a new interpretation – as an action of D-branes in the static gauge. What was called ‘Dirac-Born-Infeld’ action [10] was derived by applying the conformal invariance conditions approach of [21] in the case of mixed (Dirichlet and Neumann) boundary conditions.⁶ The same action can be also easily obtained [29] using the path integral approach as in [6].⁷ The path integral approach makes T-duality covariance properties of the resulting D-brane actions transparent, implying that all $p < 9$ brane actions can be obtained by direct dimensional reduction from the $D = 10$ ($p = 9$ brane) Born-Infeld action. Indeed, the DBI action is

⁵ Apart from the suggestion in [23] (based on type I – heterotic string duality and conjecture about special supersymmetry properties of the Born-Infeld action) that BI action may be summing up F^{2n+2} string n -loop corrections in the heterotic string theory.

⁶ Some subtleties in the approach of [10] and attempts of generalization to the non-abelian case were discussed in [28].

⁷ Here the aim is to compute the string path integral on the disc in the presence of a D -brane. This is the partition function of virtual open strings with mixed Dirichlet-Neumann boundary conditions (i.e. with ends attached to a hyperplane) propagating in a condensate of massless vector string modes. The collective coordinates X^i and internal vector A_m degrees of freedom of the D-brane are represented by the boundary background couplings as in [26,10].

not a new action, but is simply the reduction of the BI action. In particular, all solutions of the DBI action can thus be obtained from the solutions of the higher-dimensional BI action (see [9]).

Thus the form of the D-brane action is determined by the abelian $D = 10$ open string effective action [30,29] and is given by the Born-Infeld action for the $D = 10$ vector potential⁸ $A_\mu = (A_m, A_s = TX_s)$ reduced down to $p + 1$ dimensions⁹

$$\begin{aligned} S_p &= \mathcal{T}_p \int d^{p+1}x \sqrt{-\det(\eta_{\mu\nu} + T^{-1}F_{\mu\nu})} \\ &= \mathcal{T}_p \int d^{p+1}x \sqrt{-\det(\eta_{mn} + \partial_m X^s \partial_n X^s + T^{-1}F_{mn})} . \end{aligned} \quad (2.3)$$

This ‘T-duality’ relation suggests that supersymmetrization of the DBI action (and its non-abelian generalization) in flat space¹⁰ should also be determined by that of the Born-Infeld action.

Originating from the BI action, the DBI action implies similar ‘maximal field strength’ constraints on allowed physical configurations. In particular, the action for D0-brane is simply that of a relativistic particle $\int dx_0 \sqrt{1 - (\partial_0 X^s)^2}$, and the ‘maximal field strength’ constraint here is simply the standard relativistic constraint on particle’s velocity [30]. Here it is interesting to recall that it was the analogy with the square root structure of the relativistic particle action that was one of the original motivations of Born in looking for a non-linear electrodynamics action [25] which does not allow the electric field of a point charge to become infinite.

3. Some properties of bosonic $D = 4$ Born-Infeld action

The $D = 4$ Born-Infeld Lagrangian

$$L_{\text{BI}} = \sqrt{-\det_4(\eta_{mn} + F_{mn})} - 1 , \quad (3.1)$$

⁸ Here we use the Minkowski signature and the following notation: $\mu, \nu = 0, 1, \dots, 9$; $m, n = 0, 1, \dots, p$; $s, r = p + 1, \dots, 9$, $T^{-1} = 2\pi\alpha'$, $\epsilon^{mncd}\epsilon_{mncd} = -4!$. The functions A_m and X_s depend only on $x_n = (x_0, \dots, x_p)$.

⁹ In the low-energy or ‘non-relativistic’ approximation, i.e. to the leading quadratic order in F_{mn} , this action is the same as the dimensional reduction of the $D = 10$ $U(1)$ Maxwell action for A_m [31]. A simple determinant identity shows that this is true in general for the whole BI action.

¹⁰ This relation may no longer apply in the case of a non-trivial closed string background without simple isometry properties.

where we set the fundamental (scale)²($=T^{-1} = 2\pi\alpha'$ in the string theory context) equal to 1, has several remarkable features, including electro-magnetic duality and causal propagation (see, e.g., [32,33] and refs. there). Since in four dimensions

$$-\det_4(\eta_{mn} + F_{mn}) = 1 + \frac{1}{2}F_{mn}F^{mn} - \frac{1}{16}(F_{mn}F^{*mn})^2, \quad F^{*mn} \equiv \frac{1}{2}\epsilon^{mncd}F_{cd}, \quad (3.2)$$

L_{BI} interpolates between the Maxwell Lagrangian $\frac{1}{4}F_{mn}F^{mn}$ for small F and the total derivative (topological density) $\frac{i}{4}F_{mn}F^{*mn}$ for large F . In Euclidean signature one finds [9]

$$L_{\text{BI}} = \sqrt{(1 + \frac{1}{4}F_{mn}F^{*mn})^2 + \frac{1}{4}(F_{mn} - F_{mn}^*)^2} - 1 \geq \frac{1}{4}F_{mn}F^{*mn}, \quad (3.3)$$

implying that the minimum of the Euclidean action is attained at (abelian) self-dual fields (for a discussion of related BPS bounds for DBI actions see [34]). Another useful representation is

$$L_{\text{BI}} = \sqrt{(1 + I_2)^2 + 2I_4} - 1 = I_2 + I_4[1 + O(F^2)], \quad (3.4)$$

$$I_2 \equiv \frac{1}{4}F_{mn}F^{mn}, \quad I_4 \equiv -\frac{1}{8}\left[F_{mn}F^{nk}F_{kl}F^{lm} - \frac{1}{4}(F_{mn}F^{mn})^2\right] = -\frac{1}{8}(F^{(+)})^2(F^{(-)})^2.$$

That $L_{\text{BI}} = I_2 + I_4 + O(F^6)$ is true in all dimensions, but it is only in $D = 4$ that all higher order terms are proportional to I_4 . This fact is reflected in the structure of the supersymmetric generalization of the $D = 4$ BI action (see section 4.1).

The $D = 4$ Born-Infeld action is obviously symmetric under $F \leftrightarrow F^*$ and is also covariant under the electric-magnetic (or vector \rightarrow vector) duality, as can be concluded from the structure of the equations of motion [32] (see also [35,36]) or demonstrated directly at the level of the action by following the standard steps of introducing the Lagrange multiplier for the $F = dA$ constraint and solving for F in the classical approximation [29].

In four dimensions it is possible to write down the BI action in the form quadratic in F_{mn} by introducing two complex auxiliary scalar fields [37]. First, we replace L_{BI} (changing its overall sign as appropriate for the Minkowski signature) by

$$-\sqrt{-\det(\eta_{mn} + F_{mn})} \rightarrow -\frac{1}{2}V \det(\eta_{mn} + F_{mn}) + \frac{1}{2}V^{-1},$$

use (3.2) and introduce the second auxiliary field U to ‘split’ the quartic $(FF^*)^2$ term. Finally, we can eliminate the term with V^{-1} by introducing a complex auxiliary scalar $a = a_1 + ia_2$, $\bar{a} = a_1 - ia_2$, and writing the Lagrangian as [37]

$$L_4 = -\frac{1}{2}V(a + \bar{a} + \bar{a}a - \frac{1}{2}F_{mn}F^{mn}) + \frac{1}{2}U[i(a - \bar{a}) + \frac{1}{2}F_{mn}F^{*mn}] - \frac{1}{2}(a + \bar{a}), \quad (3.5)$$

or as

$$L_4 = -\text{Im} \left(\lambda \left[a + \frac{1}{2} \bar{a} a - \frac{1}{4} (F_{mn} F^{mn} + i F_{mn} F^{*mn}) \right] + i a \right), \quad (3.6)$$

$$\lambda = \lambda_1 + i \lambda_2 \equiv U + i V.$$

The constraint implied by λ is solved by $a = a(F)$ with $\text{Im } a(F) = \frac{1}{4} F^{mn} F_{mn}^*$ and the real part

$$\text{Re } a(F) = \sqrt{1 + \frac{1}{2} F^2 - \frac{1}{16} (F F^*)^2} - 1, \quad (3.7)$$

which (up to sign) is the BI Lagrangian itself. This gives a natural ‘explanation’ for the square root structure of the Born-Infeld action. One can thus view the $D = 4$ BI action as resulting from a peculiar theory for two complex non-propagating scalars (λ, a) coupled non-minimally to a vector. Shifting λ by i the Lagrangian (3.6) can be put into the form that does not contain terms linear in the fields

$$L_4 = -\frac{1}{4} F_{mn} F^{mn} + \frac{1}{2} \bar{a} a - \text{Im} \left(\lambda \left[a + \frac{1}{2} \bar{a} a - \frac{1}{4} (F_{mn} F^{mn} + i F_{mn} F^{*mn}) \right] \right). \quad (3.8)$$

Since in this form the BI action is quadratic in the vector field, it is very simple to demonstrate its covariance under the vector-vector duality. Adding the Lagrange multiplier term $\frac{1}{2} \tilde{F}^{*ab} F_{ab}$, where \tilde{F}_{ab} is the strength of the dual vector field, and integrating out F_{ab} we find that the dual action has the same form as (3.6) with ¹¹

$$F_{mn} \rightarrow \tilde{F}_{mn}, \quad \lambda \rightarrow -\lambda^{-1}, \quad a \rightarrow -i \lambda a. \quad (3.9)$$

Like the Maxwell action, the action (3.6) is not invariant under this duality. There exists, however, an equivalent action containing one extra vector field variable which is manifestly duality-symmetric [37]. duality-symmetric actions was explained

The BI Lagrangian in the form (3.6),(3.8) may be viewed as a special case of the following Lagrangian for a vector coupled non-minimally to a set of massive scalars

$$L = -\frac{1}{4} F_{mn} F^{mn} - \frac{1}{2} (\partial_a \varphi_i)^2 - \frac{1}{2} m_i^2 \varphi_i^2 + g_{ijk} \varphi_i \varphi_j \varphi_k + \varphi_i (\alpha_i F_{mn} F^{mn} + \beta_i F_{mn} F^{*mn}). \quad (3.10)$$

In the limit when masses of scalars are much larger than their gradients so that the $(\partial_a \varphi_i)^2$ terms may be ignored, (3.10) reduces to (3.8) with the scalars φ_n being linear combinations of $\lambda_1, \lambda_2, a_1, a_2$ in (3.8). This action may be viewed as a truncation of the cubic open string field theory action which reproduces the BI action as an effective action upon integrating out at the string tree level all massive string modes (represented here by φ_i) [6,38]. The kinetic terms $(\partial_a \varphi_i)^2$ may be dropped since they lead to derivative-dependent $O(\partial F)$ terms which, by definition, are not included in the leading part of the low-energy effective action. Note that to represent higher dimensional Born-Infeld action in a cubic form similar to (3.10) one would need to introduce auxiliary tensor fields to ‘split’ the higher-order F^k invariants in $\det(\eta_{mn} + F_{mn})$.

¹¹ The equations of motion derived from the vector terms in the action (3.6) have the full $SL(2, R)$ invariance: $\lambda \rightarrow \frac{p\lambda + q}{k\lambda + l}$, $F_{mn} \rightarrow (kU + l)F_{mn} + kV F_{mn}^*$, $pl - qk = 1$, see also [35,36].

4. Supersymmetric Born-Infeld actions with manifest $D = 4$ supersymmetry

Below we shall review what is known about generalizations of Born-Infeld action with manifest $D = 4$ supersymmetry.

There exists a remarkable connection between (i) partial supersymmetry breaking, (ii) nonlinear realizations of extended supersymmetry, (iii) BPS solitons, and (iv) nonlinear Born-Infeld type actions (see, e.g., [39,40,41,37] and refs. there). Extended $\mathcal{N} > 1$ supersymmetry can be partially broken either by a translationally non-invariant background (soliton) in a second-derivative higher-dimensional theory or by a translationally invariant vacuum in a nonrenormalizable theory in four dimensions containing non-minimal interactions [42].

The interpretation (and derivation) of $\mathcal{N} = 1$ supersymmetric BI action [12,13] as the action for a Goldstone multiplet associated with partial breaking of $\mathcal{N} = 2$ to $\mathcal{N} = 1$ supersymmetry was suggested in [41]. As was demonstrated in [37], the connection between partial breaking of supersymmetry and nonlinear actions is not accidental and has to do with constraints that lead directly to nonlinear actions of Born-Infeld type. Spontaneously broken symmetries give nonlinear realizations of the broken symmetry group. A standard way to find such realizations is to begin with a linear representation and impose a nonlinear constraint. The constrained superfield approach [43,37] appears to be a universal and transparent way of deriving and dealing with these actions.

In a similar way, a massless $\mathcal{N} = 2$ vector multiplet may be also considered as a Goldstone multiplet associated with partial spontaneous breaking of $\mathcal{N} = 4$ supersymmetry to $\mathcal{N} = 2$ [44]. The $\mathcal{N} = 2$ analog of the $\mathcal{N} = 1$ supersymmetric Born-Infeld action was suggested in [45]. Though this was not proved to all orders, it is likely that the bosonic part of this action is related (after a field redefinition eliminating higher derivative scalar terms) to the DBI action for a 3-brane moving in 6 dimensional space-time (with two scalars of the $\mathcal{N} = 2$ vector multiplet playing the role of the transverse collective coordinates).

The $\mathcal{N} = 4$ supersymmetric extension of the Born-Infeld action (written, e.g., in terms of $\mathcal{N} = 1$ or $\mathcal{N} = 2$ superfields) is not known at present and it appears to be non-trivial to construct it (cf. [37]). Below we shall describe what can be learned about the structure of the F^4 term in such $\mathcal{N} = 4$ action using the knowledge of the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ Born-Infeld actions and assuming the expected global $SU(3)$ symmetry of the $\mathcal{N} = 4$ action written in terms of $\mathcal{N} = 1$ superfields [46].

4.1. $\mathcal{N} = 1$ supersymmetric action

The $\mathcal{N} = 1$ supersymmetric BI action can be written in the following way [13]¹²

$$S = \frac{1}{2} \int d^4x \left(\int d^2\theta W^\alpha W_\alpha + h.c. \right) + \int d^4x \int d^2\theta d^2\bar{\theta} B(K, \bar{K}) W^\alpha W_\alpha \bar{W}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}} , \quad (4.1)$$

where

$$B \equiv \frac{1}{1 - \frac{1}{2}(K + \bar{K}) + \sqrt{1 - (K + \bar{K}) + \frac{1}{4}(K - \bar{K})^2}} = \frac{1}{2} + \frac{1}{4}(K + \bar{K}) + \dots , \quad (4.2)$$

$$K \equiv D^2(W^\alpha W_\alpha) , \quad \bar{K} \equiv \bar{D}^2(\bar{W}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}}) . \quad (4.3)$$

The action depends in general on dimensional scale parameter which is set to 1 as in (3.1). Since

$$(D^2 W^2)_{\theta, \bar{\theta}=0} = -\frac{1}{4} F^{mn} F_{mn} - \frac{i}{4} F^{mn} F_{mn}^* - \frac{i}{2} \lambda \sigma^m \partial_m \bar{\lambda} + \frac{1}{2} D^2$$

the bosonic part of the action depends on the square of the auxiliary field D so that $D = 0$ is always a solution.

To get insight into the structure of (4.1) and to exhibit its invariance under the second (spontaneously broken, i.e. non-linearly realized) supersymmetry [41] it is useful to rederive this non-linear action using constrained $\mathcal{N} = 2$ superfield approach [37]. We start with $\mathcal{N} = 2$ vector multiplet described by constrained chiral field strength $\mathcal{W}(x, \theta_1, \theta_2)$ that obeys the Bianchi identity $D_{ab}^2 \mathcal{W} = C_{ac} C_{bd} \bar{D}^{2cd} \bar{\mathcal{W}}$ ($a, b = 1, 2$). Defining $D \equiv D_1$, $Q \equiv D_2$ this becomes $D^2 \mathcal{W} = \bar{Q}^2 \bar{\mathcal{W}}$, $DQ\mathcal{W} = -\bar{D}\bar{Q}\bar{\mathcal{W}}$. We break $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$ by assuming that \mathcal{W} has a Lorentz-invariant condensate $\langle \mathcal{W} \rangle$ (we set the scale of the supersymmetry breaking to 1):

$$\mathcal{W} \rightarrow \langle \mathcal{W} \rangle + \mathcal{W} , \quad \langle \mathcal{W} \rangle = -\theta_2^2 , \quad \langle Q^2 \mathcal{W} \rangle = 1 , \quad D\langle \mathcal{W} \rangle = 0 . \quad (4.4)$$

We reduce the field content to a single $\mathcal{N} = 1$ superfield by imposing¹³

$$\mathcal{W}^2 = 0 . \quad (4.5)$$

¹² We follow the same conventions as in [37], in particular, $D^2 = \frac{1}{2} D^\alpha D_\alpha$.

¹³ This removes independent chiral superfield part of the $\mathcal{N} = 2$ multiplet. It would be interesting to generalize the discussion to the case when the chiral superfield is first kept independent and then integrated out.

Then the above constraints imply

$$Q^2\mathcal{W} = \bar{D}^2\bar{\mathcal{W}} - 1, \quad 0 = \frac{1}{2}Q^2\mathcal{W}^2 = \mathcal{W}(\bar{D}^2\bar{\mathcal{W}} - 1) + \frac{1}{2}Q^\alpha\mathcal{W}Q_\alpha\mathcal{W}. \quad (4.6)$$

Projecting to $\mathcal{N} = 1$ superspace by setting $\theta_2 = 0$ and defining the $\mathcal{N} = 1$ superfields

$$\Phi \equiv \mathcal{W}|_{\theta_2=0}, \quad W_\alpha \equiv -Q_\alpha\mathcal{W}|_{\theta_2=0}, \quad (4.7)$$

we find that the constraint for the chiral superpartner of the vector multiplet in the $\mathcal{N} = 1$ superspace description of the $\mathcal{N} = 2$ vector multiplet is

$$\Phi = \Phi\bar{D}^2\bar{\Phi} + \frac{1}{2}W^\alpha W_\alpha, \quad (4.8)$$

which coincides with the constraint in [41].

Because of the constraints (4.4),(4.5),(4.8), there are many equivalent $\mathcal{N} = 2$ forms that all give the same $\mathcal{N} = 1$ action [37]. One example is a class of actions proportional to the $\mathcal{N} = 2$ Fayet-Iliopoulos term: $\int d^2\theta_1 d^2\theta_2 \mathcal{F}(\mathcal{W}) = \mathcal{F}''(0) \int d^2\theta_1 \Phi$. Another action that leads to the same constraints and the final $\mathcal{N} = 1$ Born-Infeld action is the standard free $\mathcal{N} = 2$ vector action, i.e. the action for the $\mathcal{N} = 1$ vector (V) and chiral (Φ) superfields, plus a term with a chiral $\mathcal{N} = 1$ superfield Lagrange multiplier Λ imposing the constraint (4.8)

$$S = \int d^4x \left(\int d^2\theta \left[\left(\frac{1}{2}W^\alpha W_\alpha + \Phi\bar{D}^2\bar{\Phi} \right) + i\Lambda \left(\frac{1}{2}W^\alpha W_\alpha + \Phi\bar{D}^2\bar{\Phi} - \Phi \right) \right] + h.c. \right). \quad (4.9)$$

Shifting $\Lambda \rightarrow \Lambda + i$, we get

$$S = \int d^4x \left[\int d^2\theta \left(i\Lambda \left[\frac{1}{2}W^\alpha W_\alpha + \Phi\bar{D}^2\bar{\Phi} - \Phi \right] + \Phi \right) + h.c. \right]. \quad (4.10)$$

The resulting action is thus simply

$$S = \int d^4x \left[\int d^2\theta \Phi(W, \bar{W}) + h.c. \right], \quad (4.11)$$

where Φ is the solution of the constraint (4.8). Since the explicit solution of (4.8) is [41]

$$\Phi(W, \bar{W}) = \frac{1}{2}W^\alpha W_\alpha + \frac{1}{2}\bar{D}^2 \left[B(K, \bar{K}) W^\alpha W_\alpha \bar{W}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}} \right], \quad (4.12)$$

where B was defined in (4.2), the action (4.11) is nothing but the $\mathcal{N} = 1$ supersymmetric Born-Infeld action (4.1).

One concludes [41,37] that the requirement of partially broken $\mathcal{N} = 2$ supersymmetry uniquely fixes the action for the $\mathcal{N} = 1$ vector multiplet to be the supersymmetric Born-Infeld action. As explained in [37], the $\mathcal{N} = 1$ supersymmetric Born-Infeld action also emerges as an effective action from the $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ supersymmetry breaking model of [42] when one decouples (‘integrates out’) the massive chiral multiplet.

The bonus of the above derivation is that it reveals the hidden non-linearly realized supersymmetry of the Born-Infeld action (4.11) – the broken half of the original $\mathcal{N} = 2$ symmetry [41]. The second ($\mathcal{N} = 2$) supersymmetry transformation law follows from the above constraints and definitions of the $\mathcal{N} = 1$ superfield components,

$$\delta_2 \Phi \equiv (\eta^\alpha Q_\alpha + \bar{\eta}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}) \mathcal{W}|_{\theta_2=0} = -\eta^\alpha W_\alpha, \quad \delta_2 W_\alpha = \eta_\alpha (\bar{D}^2 \bar{\Phi} - 1) - i \bar{\eta}^{\dot{\alpha}} \partial_{\alpha \dot{\alpha}} \Phi. \quad (4.13)$$

Note that this transformation is non-linear since Φ (4.12) contains terms of all orders in W (and thus also in the fundamental scale parameter or in $2\pi\alpha'$).

The bosonic part of the supersymmetric action (4.10) is exactly the BI action represented in the form (3.6) with the two auxiliary complex scalar fields a and λ being the corresponding scalar components of the chiral superfields Φ and Λ in (4.10). As in the bosonic case (3.6), the Lagrange multiplier representation (4.10) of the action also simplifies [37] the proof [41] of the duality covariance of the $\mathcal{N} = 1$ supersymmetric Born-Infeld action (4.11),(4.12).

Let us make a brief comment about the quantum properties of the Born-Infeld actions. Viewing BI action as a leading term in the low-energy effective action of string theory, it does not make much sense to quantize it directly,¹⁴ unless one is systematically keeping all momenta small compared to the cutoff $\frac{1}{\sqrt{\alpha'}}$ as in other effective field theories (see, e.g., [47]). Still, formally, one may try to view the BI action as defining a fundamental theory and compute the corresponding quantum corrections using, e.g., background field method. It is easy to see that *logarithmically* divergent corrections to the abelian bosonic BI action will involve derivatives of the field strength, i.e. the original BI action is not renormalizable. The same is true in the $\mathcal{N} = 1$ supersymmetric case: as follows from (4.1), all terms additional to the Maxwell W^2 term in the $\mathcal{N} = 1$ BI Lagrangian are no

¹⁴ Ignoring derivative corrections, Born-Infeld action represents a sum of string tree diagrams with massive modes on internal lines and massless vectors on external lines. Quantum loop corrections to Born-Infeld action thus represent only a subclass of all string loop diagrams where, e.g., loops of massive modes are not included. It is only the sum of all string diagrams at a given loop order that is expected to be UV finite.

longer F-terms, but D-terms, and thus may be deformed by quantum corrections.¹⁵ This was indeed confirmed by explicit computations in [48,49] which demonstrated that the leading 1-loop logarithmically divergent correction to the ($\mathcal{N} \geq 2$) supersymmetric Born-Infeld action has the $\partial^4 F^4$ form. It is not completely surprising that the same 4-derivative term (whose $\mathcal{N} = 1$ structure $\sim \int d^4\theta \partial_m W^\alpha \partial^m W_\alpha \partial_n \bar{W}^{\dot{\alpha}} \partial^n \bar{W}_{\dot{\alpha}}$ [49] is determined by the supersymmetry) appears as the leading derivative correction to the Born-Infeld term in the tree-level open superstring effective action [20] (see section 7).

4.2. $\mathcal{N} = 2$ supersymmetric action

The $\mathcal{N} = 2$ extension of the Born-Infeld action suggested in [45] is similar in structure to the $\mathcal{N} = 1$ one (4.1):

$$S = \frac{1}{2} \int d^4x \left[\left(\int d^4\theta \mathcal{W}^2 + c.c. \right) + \frac{1}{4} \int d^4\theta d^4\bar{\theta} \mathcal{B}(\mathcal{K}, \bar{\mathcal{K}}) \mathcal{W}^2 \bar{\mathcal{W}}^2 \right], \quad (4.14)$$

where

$$\mathcal{B} = \frac{1}{1 - \frac{1}{2}(\mathcal{K} + \bar{\mathcal{K}}) + \sqrt{1 - (\mathcal{K} + \bar{\mathcal{K}}) + \frac{1}{4}(\mathcal{K} - \bar{\mathcal{K}})^2}}, \quad (4.15)$$

$$\mathcal{K} \equiv \frac{1}{2} D^4 \mathcal{W}^2, \quad \bar{\mathcal{K}} \equiv \frac{1}{2} \bar{D}^4 \bar{\mathcal{W}}^2. \quad (4.16)$$

The Lagrangian here $H = \mathcal{W}^2 + \dots$ (cf. (4.11)) satisfies the $\mathcal{N} = 2$ generalization of the $\mathcal{N} = 1$ non-linear constraint (4.8),

$$H = \frac{1}{4} H \bar{D}^4 \bar{H} + \mathcal{W}^2, \quad H^2 = 0. \quad (4.17)$$

The analogy with the $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ supersymmetry breaking case discussed above suggests a relation to the $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$ supersymmetry breaking [44] and thus the interpretation of (4.14) as the unique action for the $\mathcal{N} = 2$ vector multiplet as a Goldstone multiplet associated with the partial breaking of $\mathcal{N} = 4$ supersymmetry. In this case (4.14) should have hidden invariance under two extra spontaneously broken and non-linearly

¹⁵ Note, however, that there should be no finite quantum renormalization of the coefficient in front of the $\mathcal{N} = 1$ BI action (4.1) as part of the full quantum effective action: assuming that the second spontaneously broken supersymmetry survives at the quantum level, it should again relate the coefficient of the D-term in (4.1) to that of the F-term and thus should rule out its finite renormalization. This should be related to expected non-renormalization of the BPS 3-brane tension.

realized supersymmetries which should unambiguously determine the form of the action via the non-linear constraint (4.17).

This action contains terms without derivatives and with higher derivatives of the complex scalar field. At first sight, this seems to contradict its possible interpretation as a DBI action (the Nambu-type actions for the transverse collective coordinates should contain scalars only through their first derivatives as required by the translational invariance). However, it is likely that the higher-derivative terms can be eliminated by field redefinitions (cf. [45,50,37]). These, however, will make $\mathcal{N} = 2$ supersymmetry of the resulting action non-manifest. This clash between the requirement of dependence on first derivatives of scalars and manifest extended supersymmetry is likely to be the general property of the $\mathcal{N} \geq 2$ supersymmetric Born-Infeld actions.

To demonstrate that such unwelcome terms can indeed be redefined away let us consider the first subleading four-field term in the action (4.14) and show that the second-derivative scalar terms there are indeed proportional to the leading-order equation of motion $\partial^2\phi$ (i.e. vanish on shell) and thus can be eliminated by a field redefinition. The $\mathcal{N} = 2$ chiral superfield \mathcal{W} satisfies the constraints $\bar{D}_{\dot{\alpha}i}\mathcal{W} = 0$, $D^4\mathcal{W} = \partial^2\bar{\mathcal{W}}$, implying the following expansion in terms of $\mathcal{N} = 1$ superfields:

$$\begin{aligned}\mathcal{W} &= \Phi(\tilde{y}, \theta) + \sqrt{2}\theta_2^\alpha W_\alpha(\tilde{y}, \theta) + \theta_2^\alpha \theta_{2\alpha} \bar{D}^2 \bar{\Phi}(\tilde{y}, \theta) \\ &= \Phi(y, \theta) + i\theta_2 \sigma^m \bar{\theta}_2 \partial_m \Phi(y, \theta) + \frac{1}{4}\theta_2^2 \bar{\theta}_2^2 \partial^2 \Phi + \sqrt{2}\theta_2^\alpha W_\alpha(y, \theta) - \frac{i}{\sqrt{2}}\theta_2^2 \partial_m W \sigma^m \bar{\theta}_2 + \theta_2^2 \bar{D}^2 \bar{\Phi}(y, \theta),\end{aligned}\tag{4.18}$$

where $\theta \equiv \theta_1$ and $\tilde{y} = y + i\theta_2 \sigma \bar{\theta}_2 = x + i\theta \sigma \bar{\theta} + i\theta_2 \sigma \bar{\theta}_2$. Φ has the standard $\mathcal{N} = 1$ component expansion

$$\Phi = \varphi + i\theta \sigma^m \bar{\theta} \partial_m \varphi + \frac{1}{4}\theta^2 \bar{\theta}^2 \partial^2 \varphi + \sqrt{2}\theta \psi - \frac{i}{\sqrt{2}}\theta^2 \partial_m \psi \sigma^m \bar{\theta} + \theta^2 \mathcal{F}.\tag{4.19}$$

The leading correction term in (4.14) is

$$\mathcal{I}_4 = \int d^4\theta \mathbf{I}_4, \quad \mathbf{I}_4 = \int d^4\theta_2 \mathcal{W}^2 \bar{\mathcal{W}}^2,\tag{4.20}$$

where \mathbf{I}_4 can be expressed in terms of $\mathcal{N} = 1$ fields as follows

$$\begin{aligned}\mathbf{I}_4 &= W^\alpha W_\alpha \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} + i(\bar{\Phi} \partial_m \Phi - \Phi \partial_m \bar{\Phi}) W^\alpha \sigma_{\alpha\dot{\alpha}}^m \bar{W}^{\dot{\alpha}} \\ &\quad + \frac{1}{2}(\bar{\Phi} \bar{\Phi} \partial_m \Phi \partial_m \Phi + \Phi \Phi \partial_m \bar{\Phi} \partial_m \bar{\Phi}) - 2\Phi \bar{\Phi} \partial_m \Phi \partial_m \bar{\Phi}.\end{aligned}\tag{4.21}$$

The cross-term can be written also as $W^\alpha \bar{W}_{\dot{\alpha}} D_\alpha \Phi \bar{D}^{\dot{\alpha}} \bar{\Phi}$. Related expressions for the $\mathcal{N} = 2$ invariant $\mathcal{W}^2 \bar{\mathcal{W}}^2$ in terms of $\mathcal{N} = 1$ superfields appeared in [50,45]. If one is allowed to integrate by parts (which is possible in the action under the integral over x -space) and omit terms proportional to equations of motion (which can be redefined away by a transformation preserving $\mathcal{N} = 1$ supersymmetry as in [50]) then the Φ^4 terms in (4.21) reduce to just one term only, since

$$-\partial_m(\Phi^2)\partial_m(\bar{\Phi}^2) + \frac{1}{4}\partial^\mu\partial_\mu(\Phi^2\bar{\Phi}^2) = 2\partial_m(\Phi\bar{\Phi})\partial_m(\Phi\bar{\Phi}) - \frac{3}{4}\partial^\mu\partial_\mu(\Phi^2\bar{\Phi}^2) . \quad (4.22)$$

Integrating over θ one finds that the component form of the Φ^4 terms in (4.21) agrees with the 4-derivative term $(\partial\varphi)^2(\partial\bar{\varphi})^2$ in the non-linear action for a chiral multiplet in [51,37].¹⁶

To all orders the scalar part of the $\mathcal{N} = 2$ action is expected to coincide (after field redefinitions) with

$$L = \sqrt{\det(\delta_{mn} + \partial_m X^1 \partial_n X^1 + \partial_m X^2 \partial_n X^2)}$$

which can be written in the form ($\varphi \equiv X^1 + iX^2$) [37]

$$\begin{aligned} L(\varphi) &= \sqrt{1 + \partial\varphi\partial\bar{\varphi} + \frac{1}{4}(\partial\varphi\partial\bar{\varphi})^2 - \frac{1}{4}(\partial\varphi)^2(\partial\bar{\varphi})^2} \\ &= 1 + \frac{1}{2}\partial\varphi\partial\bar{\varphi} - \frac{\frac{1}{4}(\partial\varphi)^2(\partial\bar{\varphi})^2}{1 + \frac{1}{2}\partial\varphi\partial\bar{\varphi} + \sqrt{(1 + \frac{1}{2}\partial\varphi\partial\bar{\varphi})^2 - \frac{1}{4}(\partial\varphi)^2(\partial\bar{\varphi})^2}} . \end{aligned} \quad (4.23)$$

This is the bosonic part of the action of the non-linear chiral multiplet [37,52,53] (dual to a tensor multiplet action [52,37,53]) with $\mathcal{N} = 1$ superfield Lagrangian

$$L(\Phi) = \Phi\bar{\Phi} + \frac{\frac{1}{2}(D^\alpha\Phi D_\alpha\Phi)(\bar{D}^{\dot{\alpha}}\bar{\Phi}\bar{D}_{\dot{\alpha}}\bar{\Phi})}{1 + A + \sqrt{(1 + A)^2 - B}} , \quad (4.24)$$

$$A \equiv \partial^{\alpha\dot{\alpha}}\Phi\partial_{\alpha\dot{\alpha}}\bar{\Phi} , \quad B \equiv (\partial^{\alpha\dot{\alpha}}\Phi\partial_{\alpha\dot{\alpha}}\Phi)(\partial^{\alpha\dot{\alpha}}\bar{\Phi}\partial_{\alpha\dot{\alpha}}\bar{\Phi}) .$$

The Lagrangian (4.24) has manifest translational symmetry and thus defines the action representing the 3-brane in 6 dimensions of ref. [40]. It is natural to expect that there exists an exact $\mathcal{N} = 1$ superfield redefinition that puts the Φ -dependent part of the $\mathcal{N} = 2$ BI action (4.14) into the form (4.24).

¹⁶ Note that the expansion of the full bosonic action is

$$L = 1 + \frac{1}{2}\partial_m\varphi\partial_m\bar{\varphi} + \frac{1}{4}F^2 - \frac{1}{8}[F^4 - \frac{1}{4}(F^2)^2] - \frac{1}{2}(F_{mn}^2 - \frac{1}{4}F^2\delta_{mn})\partial_m\varphi\partial_n\bar{\varphi} - \frac{1}{8}\partial_m\varphi\partial_m\varphi\partial_n\bar{\varphi}\partial_n\bar{\varphi},$$

where, modulo a total derivative, $\partial_m\varphi\partial_m\varphi\partial_n\bar{\varphi}\partial_n\bar{\varphi} = 2(\partial_m\varphi\partial_m\bar{\varphi})^2$.

4.3. Four-derivative terms in the $\mathcal{N} = 4$ supersymmetric Born-Infeld action

It would be interesting for several reasons (e.g., in connection with quantum properties of D3-branes and their comparison with supergravity) to find a manifestly supersymmetric formulation of $\mathcal{N} = 4$ Born-Infeld action generalizing $\mathcal{N} = 4$ Maxwell theory. Related (by a field redefinition) component action with 4 linearly realized and 4 nonlinearly realized global $D = 4$ supersymmetries can be found by fixing static gauge and κ symmetry gauge in the D3-brane action with global $D = 10$ supersymmetry constructed in [54,55,56] (see section 5). However, the form of the $\mathcal{N} = 4$ supersymmetric action with manifest undeformed linear $D = 4$ supersymmetry, e.g., written in terms of unconstrained $\mathcal{N} = 1, D = 4$ superfields – one real vector with field strength W_α and 3 chiral scalar Φ_a ($a = 1, 2, 3$), is not known. For Φ_2, Φ_3 set equal to zero the action should reduce to the $\mathcal{N} = 1$ form of the $\mathcal{N} = 2$ action (4.14). After a field redefinition eliminating higher derivative scalar terms the bosonic part of the action should become the $10 \rightarrow 4$ dimensional reduction of the $D = 10$ Born-Infeld action, i.e. the DBI action of a D3-brane moving in 10 dimensions,

$$S = \int d^4x \sqrt{-\det(\eta_{mn} + \partial_m X^s \partial_n X^s + F_{mn})}, \quad s = 1, \dots, 6, \quad (4.25)$$

or, equivalently, squaring the matrix which appears under the determinant,

$$S = \int d^4x \left[-\det(\eta_{mn} + 2t_{mn} + t_{mr}t_{nr} + F_{mr}F_{nr} + 2t_{r(m}F_{n)r}) \right]^{1/4}, \quad (4.26)$$

where $t_{mn} \equiv \partial_m X^s \partial_n X^s$.

The 6 real coordinates X^s should be related to the 3 complex scalar components of Φ_a by

$$\varphi^a = X^a + iX^{a+3}, \quad \partial_m X^s \partial_n X^s = \partial_{(m} \varphi^a \partial_{n)} \bar{\varphi}^a. \quad (4.27)$$

The bosonic Lagrangian in (4.25), (4.26) has the following expansion in powers of derivatives

$$\begin{aligned} L = 1 + \frac{1}{2} \partial_m X^s \partial_m X^s + \frac{1}{4} F^2 - \frac{1}{8} [F^4 - \frac{1}{4} (F^2)^2] - \frac{1}{2} (F_{mn}^2 - \frac{1}{4} F^2 \delta_{mn}) \partial_m X^s \partial_n X^s \\ + \frac{1}{8} [(\partial_m X^s \partial_m X^s)^2 - 2(\partial_m X^s \partial_n X^s)(\partial_m X^u \partial_n X^u)] + \dots, \end{aligned} \quad (4.28)$$

where $F_{mn}^2 = F_{mr}F_{nr}$, or, equivalently,

$$\begin{aligned} L = 1 + \frac{1}{2} \partial_m \varphi^a \partial_m \bar{\varphi}^a + \frac{1}{4} F^2 - \frac{1}{8} [F^4 - \frac{1}{4} (F^2)^2] - \frac{1}{2} (F_{mn}^2 - \frac{1}{4} F^2 \delta_{mn}) \partial_m \varphi^a \partial_n \bar{\varphi}^a \\ + \frac{1}{8} (\partial_m \varphi^a \partial_m \bar{\varphi}^a \partial_n \varphi^b \partial_n \bar{\varphi}^b - \partial_m \varphi^a \partial_m \varphi^b \partial_n \bar{\varphi}^a \partial_n \bar{\varphi}^b - \partial_m \varphi^a \partial_m \bar{\varphi}^b \partial_n \varphi^b \partial_n \bar{\varphi}^a). \end{aligned} \quad (4.29)$$

The knowledge of the bosonic action (4.29), the 4-derivative term in the $\mathcal{N} = 2$ action (4.21) and the condition of $SU(3)$ symmetry in the chiral superfield sector allows one to deduce the analog of the term (4.21) in the $\mathcal{N} = 1$ superfield form of the $\mathcal{N} = 4$ Born-Infeld action [46]. This gives the $\mathcal{N} = 4$ generalization of the $I_4 \sim F^4$ invariant in the Born-Infeld action (3.4) and its $\mathcal{N} = 1$ ($W^2\bar{W}^2$) (4.1) and $\mathcal{N} = 2$ (4.21) counterparts.

Modulo the terms with $\partial^2\Phi$ the only possible $SU(3)$ invariant generalizations of the three Φ^4 terms in (4.21) are

$$P_1 = \frac{1}{2}(\bar{\Phi}^a\bar{\Phi}^b\partial_m\Phi^a\partial_m\Phi^b + \Phi^a\Phi^b\partial_m\bar{\Phi}^a\partial_m\bar{\Phi}^b),$$

$$P_2 = \Phi^b\bar{\Phi}^b\partial_m\Phi^a\partial_m\bar{\Phi}^a, \quad P_3 = \Phi^b\bar{\Phi}^a\partial_m\Phi^a\partial_m\bar{\Phi}^b, \quad (4.30)$$

$$P_1 = -P_2 - P_3 + \frac{1}{4}\partial^m\partial_m(\Phi^a\bar{\Phi}^a\Phi^b\bar{\Phi}^b) \quad (4.31)$$

Using (4.19) and dropping the terms with $\partial^2\varphi$ which can be eliminated by a field redefinition in the total action containing $\Phi^2 + \Phi^4$ terms, as well as total derivative terms, we find that the scalar field parts of these invariants are

$$\int d^4\theta \ P_1 = 2\partial_m\varphi^a\partial_m\varphi^b\partial_n\bar{\varphi}^a\partial_n\bar{\varphi}^b, \quad (4.32)$$

$$\int d^4\theta \ P_2 = \partial_m\varphi^a\partial_m\bar{\varphi}^b\partial_n\varphi^b\partial_n\bar{\varphi}^a - \partial_m\varphi^a\partial_m\bar{\varphi}^a\partial_n\varphi^b\partial_n\bar{\varphi}^b - \partial_m\varphi^a\partial_m\varphi^b\partial_n\bar{\varphi}^a\partial_n\bar{\varphi}^b,$$

$$\int d^4\theta \ P_3 = \partial_m\varphi^a\partial_m\bar{\varphi}^a\partial_n\varphi^b\partial_n\bar{\varphi}^b - \partial_m\varphi^a\partial_m\varphi^b\partial_n\bar{\varphi}^a\partial_n\bar{\varphi}^b - \partial_m\varphi^a\partial_m\bar{\varphi}^b\partial_n\varphi^b\partial_n\bar{\varphi}^a.$$

To determine the relevant linear combination of P_1, P_2, P_3 that generalizes the Φ^4 terms in (4.21) we shall use the comparison of the scalar field terms with the corresponding $(\partial X)^4$ structures in the BI action (4.29) (which can be also obtained by dimensional reduction from the F^4 terms in the $D = 10$ Born-Infeld action). We find that the right combination is $P_1 + P_2 - 3P_3$, i.e. the $SU(3)$ invariant generalization of (4.21) to the case of several chiral superfields is

$$\mathbf{I}_4 = W^\alpha W_\alpha \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} + i(\bar{\Phi}^a\partial_m\Phi^a - \Phi^a\partial_m\bar{\Phi}^a)W^\alpha\sigma_{\alpha\dot{\alpha}}^m\bar{W}^{\dot{\alpha}} \\ + \frac{1}{2}(\bar{\Phi}^a\bar{\Phi}^b\partial_m\Phi^a\partial_m\Phi^b + \Phi^a\Phi^b\partial_m\bar{\Phi}^a\partial_m\bar{\Phi}^b) + \Phi^b\bar{\Phi}^b\partial_m\Phi^a\partial_m\bar{\Phi}^a - 3\bar{\Phi}^a\Phi^b\partial_m\Phi^a\partial_m\bar{\Phi}^b. \quad (4.33)$$

Ignoring $\partial^2\Phi$ terms, the Φ^4 terms in (4.33) can be rewritten simply as $-4P_3 + \frac{1}{4}\partial^\mu\partial_\mu(\Phi^a\bar{\Phi}^a\Phi^b\bar{\Phi}^b)$, or as

$$2\partial_m(\Phi^a\bar{\Phi}^b)\partial_\mu(\Phi^b\bar{\Phi}^a) - \frac{3}{4}\partial^\mu\partial_\mu(\Phi^a\bar{\Phi}^a\Phi^b\bar{\Phi}^b), \quad (4.34)$$

where the total derivative term may be dropped in the action. This simple $SU(3)$ invariant expression is the generalization of the $\mathcal{N} = 2$ expression (4.22).

The $\mathcal{N} = 4$ analog of the F^4 term in the Born-Infeld action may be written also in terms of $\mathcal{N} = 2$ superfields using projective superspace approach [57,58], or as an integral of the 4-th power of the analytic on-shell $\mathcal{N} = 4$ superfield of [59].

5. Supersymmetric Born-Infeld actions with ‘deformed’ supersymmetry from D-brane actions

Component D3-brane actions with $D = 10$ space-time supersymmetry and local reparametrization invariance and world-volume κ -symmetry were constructed in [54,55,56] (in flat and generic curved backgrounds).

The $\mathcal{N} = 1$, $D = 10$ supersymmetric Born-Infeld action was obtained in [55] by fixing the static gauge and a κ symmetry gauge in the D9-brane in flat type IIB background. Before gauge fixing the action depends on two $D = 10$ Majorana-Weyl spinors θ_1, θ_2 and is invariant under two global $D = 10$ supersymmetries. After κ symmetry gauge fixing by setting $\theta_2 = 0$ the action depends on the vector A_μ and the remaining Majorana-Weyl spinor $\theta_1 \equiv \Psi$,¹⁷

$$S_{10} = \int d^{10}x \sqrt{-\det(\eta_{\mu\nu} + F_{\mu\nu} - 2\bar{\Psi}\Gamma_\mu\partial_\nu\Psi + \bar{\Psi}\Gamma^\rho\partial_\mu\Psi\bar{\Psi}\Gamma_\rho\partial_\nu\Psi)} . \quad (5.1)$$

Here Γ_M are the $D = 10$ Dirac matrices. This action is invariant under the two original supersymmetries supplemented by the κ symmetry transformation to maintain the gauge. One of the resulting symmetries (corresponding to the spontaneously broken half of the $\mathcal{N} = 2$, $D = 10$ supersymmetry) is realized non-linearly. Under the other (unbroken combination of the two original symmetries) Ψ has a homogeneous transformation law. Similar action for the D3-brane is obtained by reduction to 4 dimensions [55] (cf. (4.25))

$$S_4 = \int d^4x \sqrt{-\det(\eta_{mn} + \partial_m X^s \partial_n X^s + F_{mn} + \psi_{mn})} , \quad (5.2)$$

$$\psi_{mn} \equiv 2\bar{\Psi}(\Gamma_m + \Gamma_s \partial_m X^s) \partial_n \Psi + \bar{\Psi}\Gamma^s \partial_m \Psi \bar{\Psi}\Gamma_s \partial_n \Psi .$$

The unbroken supersymmetry of (5.1) or (5.2) has complicated form with terms of all orders in F or in the fundamental scale parameter (the inverse string tension factor $2\pi\alpha'$ suppressed in (5.1)). It may be thought of as an α' -deformation of the linear supersymmetry transformations of the $\mathcal{N} = 1$, $D = 10$ Maxwell multiplet. This is opposite [55] to what was the case in the $\mathcal{N} = 1$, $D = 4$ superfield action discussed in section 4.1 where the unbroken supersymmetry was undeformed and was simply the original Maxwell supersymmetry while the broken supersymmetry was non-linear (cf. (4.13) [41]) and contained terms of all orders in F or in α' . The two formulations are presumably related by a field redefinition.

¹⁷ The form of the action in a different κ -symmetry gauge – Killing gauge – was given in [48].

The leading terms in the expansion of (5.1) (or its dimensional reduction (5.2)) should be related by a field redefinition in the fermionic sector to the known $F^2 + \alpha'^2 F^4$ deformation of the $\mathcal{N} = 1$, $D = 10$ Maxwell action [14,15]. The latter can be derived by a supersymmetric completion of the bosonic gauge theory term $F^4 - \frac{1}{4}(F^2)^2$ starting with the standard the $\mathcal{N} = 1$, $D = 10$ SYM supersymmetry transformation laws and deforming the latter by α'^2 corrections. Alternative route is directly deducing the 4-fermion terms from the open superstring amplitude (the structure of the corresponding invariant is dictated by the standard massless mode superstring 4-point kinematic factor [60]). In terms of the $D = 10$ gauge field strength $F_{\mu\nu}$ and $D = 10$ Majorana-Weyl spinor Ψ one finds (up to a field redefinition) the following supersymmetric completion of the $F^2 + F^4$ terms ($2\pi\alpha' = 1$) [15]¹⁸

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{i}{2}\bar{\Psi}\Gamma^\mu\partial_\mu\Psi + \frac{1}{8}\left[F_{\mu\nu}F^{\nu\kappa}F_{\kappa\lambda}F^{\lambda\mu} - \frac{1}{4}(F^{\mu\nu}F_{\mu\nu})^2\right. \\ \left.+ \frac{1}{2}i\bar{\Psi}\Gamma^{\mu\nu\kappa}\Psi F_{\mu\lambda}\partial^\lambda F_{\nu\kappa} + 2i\bar{\Psi}\Gamma^\mu\partial_\nu\Psi F_{\mu\lambda}F^{\lambda\nu} - \frac{1}{3}\bar{\Psi}\Gamma^\mu\partial^\nu\Psi\bar{\Psi}\Gamma_\mu\partial_\nu\Psi\right]. \quad (5.3)$$

Comparing the action (5.2) to the actions with manifest linear $\mathcal{N} > 1$ supersymmetry discussed in sections 4.2, 4.3 we see that while (5.2) depends only on the first derivatives of the scalars, the manifestly supersymmetric actions like (4.14), (4.21), (4.33) involve zero and second derivatives of the scalars before one makes field redefinitions. Such field redefinitions should be (at least partially) responsible for a non-linear modification of the supersymmetry transformation laws of the resulting translationally invariant actions like (5.2).

Let us briefly mention that one can also obtain a similar action for a D3-brane moving in curved $AdS_5 \times S^5$ vacuum background of type IIB theory [61].¹⁹ The bosonic part of this action or the action for a D3-brane moving near the core of another D3-brane has the form

$$S = \int d^4x |X|^4 \left[\sqrt{-\det(\eta_{mn} + Q|X|^{-4}\partial_m X^s \partial_n X^s + Q^{1/2}|X|^{-2}F_{mn})} - 1 \right] + S_{CS}(X), \quad (5.4)$$

¹⁸ The corresponding non-abelian expression [14] is found by taking the fields to be $U(N)$ matrices and adding symmetrized trace.

¹⁹ The space-time supersymmetric and κ -symmetric D3-brane action was constructed in terms of the invariant Cartan one-forms defined on the coset superspace $SU(2, 2|4)/[SO(4, 1) \otimes SO(5)]$. The method used is conceptually very close to the one used in [55] to find the action of a D3-brane propagating in flat space as a $D = 4$ ‘Born-Infeld plus Wess-Zumino’ -type model on the flat coset superspace ($D = 10$ super Poincare group)/($D = 10$ Lorentz group).

where $Q = 4\pi N g_s \alpha'^2$, $|X|^2 \equiv X^s X^s$, $S_{CS} \sim N \int_5 \epsilon_{s_1 \dots s_6} \bar{X}^{s_1} d\bar{X}^{s_2} \wedge \dots \wedge d\bar{X}^{s_6}$, $\bar{X}^s \equiv X^s/|X|$. The supersymmetric extension of this action generalizes (5.2). This action should coincide with the leading IR, large N , part of the *quantum* $\mathcal{N} = 4$ $SU(N)$ SYM effective action obtained by keeping the $U(1)$ $\mathcal{N} = 4$ vector multiplet as an external background and integrating out massive SYM fields [62,63,64,65].²⁰ One simple test of this conjecture is the following: since the quantum $\mathcal{N} = 4$ SYM theory is conformally invariant, the resulting action should also have (spontaneously broken by scalar field background and thus non-linearly realized) conformal symmetry. The non-linear conformal invariance of the bosonic part of the static-gauge D3-brane action in AdS_5 background was indeed demonstrated in [64,66].²¹ The validity of this “quantum SYM \rightarrow BI” conjecture suggested by the supergravity – SYM correspondence should rely on the the existence of many new non-renormalization theorems.

Like the flat-space action of [54,55], the action in [61] is invariant under the 32 global supersymmetries of the $AdS_5 \times S^5$ vacuum and κ -invariant. Its conformal invariance is a consequence of the $SO(4,2) \times SO(6)$ isometry of the $AdS_5 \times S^5$ metric and is manifest (linearly realized) before the static gauge fixing. It is only after choosing the standard Poincare coordinates and fixing the static gauge and appropriate κ -symmetry gauge it will have a “SYM effective action” interpretation (details of this procedure remain to be understood). Like the flat space action (5.2), it will then have 16 linear and 16 non-linear (conformal) supersymmetries, i.e. only the $ISO(3,1) \otimes SO(5)$ and 16 supersymmetries of the original symmetry will remain manifest, but the superconformal symmetry will be realized non-linearly. While for both D3-brane actions – in flat space and in $AdS_5 \times S^5$ space – their gauge-fixed forms have only 16 linearly realized supersymmetries, the interpretation of the remaining 16 supersymmetries as conformal ones is possible only in the $AdS_5 \times S^5$ case.²² The difference between the two actions is related to the fact that while the flat space action (5.2) has explicit scale ($\sqrt{\alpha'}$), the role of such scale in

²⁰ S_{CS} which is $SO(6)$ invariant (does not depend on $|X|$) should have purely 1-loop origin.

²¹ It was conjectured in [64] (and demonstrated for the particular case when only the modulus of X_s is non-constant) that this non-linear symmetry may be fixing the structure of the action (5.4) uniquely. This seems to be unlikely since superconformal symmetry is not sufficient to restrict the form of the vector field terms, and the scalar terms should be related to the gauge theory terms by supersymmetry.

²² This interpretation seems to depend on a proper choice the κ -symmetry gauge which should be different, e.g., from the $\theta_2 = 0$ choice in [55].

the $AdS_5 \times S^5$ action is played by the modulus of the scalar field.²³ As in the flat case, the resulting action should be invariant under complicated (‘ X -deformed’) supersymmetry transformations. Examples of similar actions in lower dimensions were constructed in [67].

6. Non-abelian generalization of Born-Infeld action

The abelian Born-Infeld action represents the derivative-independent part of the open string tree level effective action. In contrast, the part of the string effective action for the non-abelian vector field which depends on the field strength but not on its covariant derivatives is not defined unambiguously since $[D_m, D_n]F_{kl} = [F_{mn}, F_{kl}]$. One natural definition of the non-abelian Born-Infeld (NBI) action suggested in [68] which will be described below is based on replacing F_{mn} in the BI action by a non-abelian field strength and adding the symmetrized trace in front of the $\sqrt{\det}$ action.

6.1. String theory considerations

This definition can be motivated from string theory as follows [68].²⁴ One starts with the path integral representation for the generating functional for the vector scattering amplitudes on the disc²⁵

$$Z(A) = \langle \text{tr } P \exp i \int d\varphi [\dot{x}^m A_m(x) - \frac{1}{2} \psi^m \psi^n F_{mn}(x)] \rangle \quad (6.1)$$

$$= \int d^D x_0 \langle \text{tr } P \exp i \int d\varphi [\dot{\xi}^m A_m(x_0 + \xi) - \frac{1}{2} \psi^m \psi^n F_{mn}(x_0 + \xi)] \rangle ,$$

where the trace is in the fundamental representation of the Chan-Paton group, $x = x_0 + \xi(\varphi)$, $0 < \varphi \leq 2\pi$ and the averaging is done with the free string propagator restricted to the boundary of the disc, i.e. with the action $\int (\xi G^{-1} \xi + \psi K^{-1} \psi)$ ($\varphi_{12} \equiv \varphi_1 - \varphi_2$, $\epsilon \rightarrow +0$)

$$G(\varphi_1, \varphi_2) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{e^{-n\epsilon}}{n} \cos n\varphi_{12} , \quad K(\varphi_1, \varphi_2) = \frac{1}{\pi} \sum_{r=1/2}^{\infty} e^{-r\epsilon} \sin r\varphi_{12} . \quad (6.2)$$

²³ In contrast to the $AdS_5 \times S^5$ one, the flat-space Born-Infeld -type D3-brane action is not, of course, related to quantum SYM theory; instead, the higher-order terms in it are interpreted as tree-level string-theory α' corrections.

²⁴ A somewhat different proposal was made in [69].

²⁵ Here P stands for the standard path ordering. As explained in [70,20], the $[A_m, A_n]$ term in F_{mn} appears from the contact terms in the supersymmetric theta-functions in the definition of the supersymmetric path ordering.

Using the radial gauge $\xi^m A_m(x_0 + \xi) = 0$, $A_m(x_0) = 0$ (see, e.g., [71]) one finds the following expansion in terms of symmetrized products of covariant derivatives of F at x_0 ,

$$\int d\varphi \dot{\xi}^m A_m(x_0 + \xi) = \int d\varphi \dot{\xi}^m \left[\frac{1}{2} \xi^n F_{nm} + \frac{1}{3} \xi^n \xi^l D_l F_{nm} + \frac{1}{8} \xi^n \xi^l \xi^s D_{(s} D_{l)} F_{nm} + \dots \right] . \quad (6.3)$$

Then

$$Z(A) = \int d^D x_0 [\mathcal{L}(F) + O(D_{(k} \dots D_{l)} F)] , \quad (6.4)$$

$$\mathcal{L}(F) = \langle \text{tr } P \exp \left[\frac{1}{2} i F_{nm} \int d\varphi (\dot{\xi}^m \xi^n + \psi^m \psi^n) \right] \rangle . \quad (6.5)$$

Dropping all symmetrized covariant derivatives leaves us with $\mathcal{L}(F)$. The path integral in (6.5) is effectively non-gaussian because of the path ordering of the $F_{nm}(x_0)(\dot{\xi}^m \xi^n)(\varphi)$ factors which is non-trivial if the matrices F_{mn} do not commute. If we further *define* the NBI Lagrangian as part of $\mathcal{L}(F)$ which does not contain commutators of F 's, or, more precisely, which is completely symmetric in all factors of F in each monomial $\text{tr}(F \dots F)$, we can then replace the trace in (6.5) by *symmetrized* trace, i.e. treat F_{mn} matrices as if they are commuting and thus drop the path ordering symbol. The resulting path integral is then computable exactly as in the abelian case [19]

$$\mathcal{L}(F) \rightarrow L_{\text{NBI}}(F) = \text{Str} \langle \exp \left[\frac{1}{2} i F_{nm} \int_0^{2\pi} d\varphi (\dot{\xi}^m \xi^n + \psi^m \psi^n) \right] \rangle \quad (6.6)$$

$$= \text{Str} \left[-\det(\eta_{mn} + T^{-1} F_{mn}) \right]^\nu ,$$

$$\nu = -\pi \int_0^{2\pi} (\dot{G}^2 - K^2) = \left(-\sum_{n=1}^{\infty} e^{-2\epsilon n} + \sum_{r=1/2}^{\infty} e^{-2\epsilon r} \right)_{\epsilon \rightarrow 0} = \frac{1}{2} , \quad (6.7)$$

and thus

$$L_{\text{NBI}}(F) = \text{Str} \sqrt{-\det(\eta_{mn} + T^{-1} F_{mn})} . \quad (6.8)$$

This NBI action represents in a sense a ‘minimal’ non-abelian extension of the abelian Born-Infeld action which is consistent with the basic requirement of tree-level string theory – overall single trace of products of field strengths as matrices in the fundamental representation. Remarkably, it reproduces exactly the $F^2 + \alpha'^2 F^4$ terms in the full non-abelian open superstring effective action²⁶

$$(2\pi\alpha')^{-2} \text{Str} \left[\sqrt{-\det(\eta_{mn} + 2\pi\alpha' F_{mn})} - I \right]$$

²⁶ The non-abelian F^4 terms were originally found in the Str-form in [18] and in the equivalent tr-form in [17].

$$\begin{aligned}
&= \text{Str} \left[\frac{1}{4} F_{mn}^2 - \frac{1}{8} (2\pi\alpha')^2 (F^4 - \frac{1}{4} (F^2)^2) + O(\alpha'^4) \right] \\
&= \text{tr} \left[\frac{1}{4} F_{mn}^2 - \frac{1}{12} (2\pi\alpha')^2 (F_{mn} F_{rn} F_{ml} F_{rl} + \frac{1}{2} F_{mn} F_{rn} F_{rl} F_{ml} \right. \\
&\quad \left. - \frac{1}{4} F_{mn} F_{mn} F_{rl} F_{rl} - \frac{1}{8} F_{mn} F_{rl} F_{mn} F_{rl}) + O(\alpha'^4) \right] .
\end{aligned} \tag{6.9}$$

It should be stressed that these F^4 terms represent the full $O(\alpha'^2)$ term in the superstring effective action, i.e. all other possible terms with covariant derivatives can be redefined away at this order [17].

In general, the full open string effective action is given by the sum of the three types of terms: (i) NBI action (6.8); (ii) F^n terms containing factors of commutators of F 's; (iii) terms with symmetrized covariant derivatives of F . While the separation of terms with symmetrized covariant derivatives is unambiguous, terms from (i) and (ii) have similar $\text{tr}(F\dots F)$ structure and their sum reduces to the abelian Born-Infeld action in the case when F 's commute. It is clear, of course, that there is no reason to expect that the NBI action (6.8) should reproduce the full string theory expression at higher than α'^2 orders (i.e. $\alpha'^4 F^6 + \dots$).²⁷ Still, the symmetrized trace action has several exceptional features and may indeed provide a good approximation to string (or D-brane) dynamics in certain situations, e.g., described by nearly commuting or nearly covariantly constant field strengths, or by BPS configurations.

6.2. Properties of the symmetrized trace action

Before discussing some properties and generalizations of the NBI action (6.8) let us make its definition more explicit. Expanding the abelian Born-Infeld Lagrangian in powers of F we may define the Lorentz tensors $C^{m_1 n_1 \dots m_{2k} n_{2k}}$ as the coefficients in²⁸

$$\sqrt{-\det(\eta_{mn} + F_{mn})} = \sum_{k=0}^{\infty} C^{m_1 n_1 \dots m_{2k} n_{2k}} F_{m_1 n_1} \dots F_{m_{2k} n_{2k}} . \tag{6.10}$$

If $F_{mn} = F_{mn}^a T_a$ where T_a are generators of the gauge group (in the fundamental representation) the non-abelian Born-Infeld action is defined by

$$\text{Str} \sqrt{-\det(\eta_{mn} I + F_{mn})} \equiv \sum_{k=0}^{\infty} d_{a_1 \dots a_{2k}} C^{m_1 n_1 \dots m_{2k} n_{2k}} F_{m_1 n_1}^{a_1} \dots F_{m_{2k} n_{2k}}^{a_{2k}} . \tag{6.11}$$

²⁷ Therefore, possible disagreements with predictions of the full string theory effective action like the one observed in [72] (where quadratic fluctuations in a constant abelian F_{mn} background were discussed both from NBI action and string theory points of view) should not be unexpected.

²⁸ Since $\det(\eta + F) = \det(\eta + F^T) = \det(\eta - F)$ the expansion of BI and thus of NBI action contains only even powers of F .

Here the totally symmetric tensors

$$d_{a_1 \dots a_p} = \text{Str}(T_{a_1} \dots T_{a_p}) \equiv \frac{1}{p!} \text{tr}(T_{a_1} \dots T_{a_p} + \text{all permutations}) \quad (6.12)$$

are the (adjoint action) invariant tensors of the gauge Lie algebra ($\sum_i d_{a_1 \dots a'_i \dots a_p} f_{a_i b}^{a'_i} = 0$, $[T_a, T_b] = f_{ab}^c T_c$).²⁹ The definition (6.11) is thus quite natural from the mathematical point of view.

For example, for $SU(2)$ one has $T_a = \sigma_a$, $T_a T_b = \delta_{ab} + i\epsilon_{abc} T_c$, so that $d_{a_1 \dots a_{2n}} = 2\delta_{(a_1 a_2 \dots a_{2n-1} a_{2n})}$. For $SU(3)$ all $d_{a_1 \dots a_{2n}}$ are expressed in terms of $d_{ab} \sim \delta_{ab}$ and d_{abc} . The simple structure of $d_{a_1 \dots a_{2n}}$ in the $SU(2)$ case allows one to write down the $SU(2)$ NBI Lagrangian in the following form: $L_{\text{NBI}}(SU(2)) = \langle \sqrt{\det(\delta_{mn} + \mathcal{F}_{mn})} \rangle$, where $\mathcal{F}_{mn} = t_a F_{mn}^a$ and the averaging is done over the free gaussian variable t_a with the rule $\langle t_a t_b \rangle = 2\delta_{ab}$, i.e. $\langle \dots \rangle \sim \int [dt] \exp(-t_a t_a) \dots$ ³⁰

The fact that under Str one can effectively treat the factors of F as commuting simplifies the analysis of the consequences of the NBI action (see [72,74]). Indeed, most of the properties of the abelian Born-Infeld action have direct non-abelian analogs in the NBI action case. In particular, one can show that: (i) covariantly constant fields $D_m F_{kl} = 0$ are solutions of the NBI equations;³¹ (ii) the NBI action has the same BPS solutions (waves, instantons, monopoles, etc.) as the YM action $\text{tr} F_{mn}^2$ [74].³² The NBI action and

²⁹ In general, for $SU(N)$ there are $N - 1$ (=rank) basic or primitive tensors $d_{a_1 \dots a_r}$ in terms of which all other $d_{a_1 \dots a_n}$ can be expressed (see, e.g., [73]). The primitive symmetric tensors define the Casimir operators $I_r = d_{a_1 \dots a_r} T^{a_1} \dots T^{a_r}$.

³⁰ Similar representation might exist for higher $SU(N)$ groups if additional primitive invariant tensors are added as coupling constants to the action for t_a (e.g., $d_{abc} t_a t_b t_c + \dots$), making it non-gaussian.

³¹ The variation of the NBI action is $\text{Str}[\sqrt{-\det H_{kl}} (H^{-1})^{mn} (D_m \delta A_n - D_n \delta A_m)]$, where $H_{mn} \equiv \eta_{mn} + F_{mn}$. Since Str (6.12) is the sum of terms with ordinary traces which have the usual cyclic symmetry, one can always put the term with variation to the right of all others. In general, for a set of matrices M_i, K one has $\text{Str}(M_1 \dots M_n K) = \text{tr}(M_{(1} \dots M_n) K)$ where $M_{(1} \dots M_n) \equiv \frac{1}{n!} \sum (M_1 \dots M_n + \text{all permutations})$.

³² See also [75] for a discussion of Bogomol'nyi relations in the abelian $\mathcal{N} = 1$ Born-Infeld theory (4.1) combined with a Higgs scalar Lagrangian (such theory may result as a certain approximation from the NBI action). Monopole solutions in the non-abelian theory combined with Higgs sector were considered in [76].

equations of motion reduce³³ on such configurations simply to the YM action and the YM equations $D_m F^{mn} = 0$. As in the abelian case (3.2),(3.3) one can show that in four (euclidean) dimensions

$$\begin{aligned} L_{\text{NBI}} &= \text{Str} \sqrt{\det_4(\delta_{mn} + F_{mn})} = \text{Str} \sqrt{1 + \frac{1}{2} F_{mn} F^{mn} + \frac{1}{16} (F_{mn} F^{*mn})^2} \\ &= \text{Str} \sqrt{(1 + \frac{1}{4} F_{mn} F^{*mn})^2 + \frac{1}{4} (F_{mn} - F_{mn}^*)^2} . \end{aligned} \quad (6.13)$$

However, this formal representation does not mean that the resulting action is expressed in terms of the two Lorentz scalars only: Str includes all possible orderings of the F_{mn} factors.³⁴

The bosonic NBI action admits straightforward supersymmetric extensions generalizing the abelian actions like (4.1) (see [79]) or (5.1). The non-abelian generalization of (5.3) or, equivalently, the supersymmetric version of the $\text{tr}(F^2 + F^4)$ terms in (6.9) invariant under α' -deformed supersymmetry was found in [14].

6.3. Non-abelian D-brane actions

The non-abelian actions for clusters of D-branes can be again obtained by dimensional reduction of the NBI action [68] (cf. (2.3)). As was argued in [31], for a system of N parallel D-branes the fields (A_m, X_s) become $U(N)$ matrices and the low-energy Maxwell action is generalized to the $D = 10$ Yang-Mills action reduced to $p+1$ dimensions. The full action including higher order terms is determined by the dimensional reduction of the open string effective action with the gauge potential components replaced by the matrix-valued fields $A_\mu = (A_m, A_s = T X_s)$. This follows directly from the non-abelian generalization of the partition function approach to the derivation of D-brane actions discussed in [29]. T-duality relates the Neumann A_s and Dirichlet A_a vertices in the exponent in

$$Z = \langle \text{tr} P \exp i \int d\varphi [\partial_\varphi x^m A_m(x) + \partial_\perp x^s A_s(x)] \rangle . \quad (6.14)$$

³³ As in the abelian case (3.3), on self-dual configuration $\det(\delta_{mn} + F_{mn})$ reduces to a perfect square $[1 + \frac{1}{4} (F_{mn})^2]^2$.

³⁴ Thus the existence of a BPS bound similar to (3.3) is not obvious [77] (cf. [78]). The inequality relation for $\text{Str} \sqrt{\det(\dots)}$ action needs, in general, a separate proof different from the abelian argument (note also that Str-action (6.11) is defined perturbatively and its global properties depend on convergence of the series). The BPS bound seems to hold at least in the $SU(2)$ case as can be seen from the ‘abelian’ representation $L_{\text{NBI}}(SU(2)) = \langle \sqrt{\det(\delta_{mn} + \mathcal{F}_{mn})} \rangle$ mentioned above (the averaging is done with a positive definite measure).

In view of the above discussion, in situations when covariant derivatives and their commutators are small, it is natural to assume that the most important part of these corrections is represented by the NBI action (6.8). Then we arrive at the following non-abelian generalization of (2.3) [68]

$$\begin{aligned} S_p &= \mathcal{T}_p \int d^{p+1}x \text{Str} \sqrt{-\det(\eta_{\mu\nu} + T^{-1}F_{\mu\nu})} \\ &= \mathcal{T}_p \int d^{p+1}x \text{Str} \left[\sqrt{-\det \mathcal{G}_{mn}} \sqrt{\det \mathcal{G}_{rs}} \right], \end{aligned} \quad (6.15)$$

$$\mathcal{G}_{mn} \equiv \eta_{mn} + \mathcal{G}^{rs}(X) D_m X_r D_n X_s + T^{-1} F_{mn}, \quad \mathcal{G}_{rs}(X) \equiv \delta_{rs} - iT[X_r, X_s]. \quad (6.16)$$

Here Str applies not to individual A_m and X_s but to the products of components of the *field strength* $F_{\mu\nu}$, i.e. $F_{rs} = -iT^2[X_r, X_s]$, $F_{mr} = TD_m X_r = T(\partial_m X_r - i[A_m, X_r])$ and $F_{mn}(A) = \partial_m A_n - \partial_n A_m - i[A_m, A_n]$. As in (2.3), we have used the simple determinant identity (which is applicable under Str). Compared to the abelian case, now there is a non-trivial extra factor of the determinant of the ‘internal metric’ $\mathcal{G}_{rs}(X) \equiv \delta_{rs} - iT[X_r, X_s]$ (which is equal to 1 if X_s commute).³⁵

There should exist a non-trivial generalization of the action (6.15) to the case when D-branes are put in a curved background. One natural suggestion is to consider background fields as functions of $X_s = x_s I + X'_s$ where x_s is the center of mass coordinate and expand in power series in $SU(N)$ coordinates X'_s . For example, for the dilaton interaction $\text{tr}(\phi F_{mn}^2)$ this leads to $\sum \frac{1}{n!} \partial_{s_1} \dots \partial_{s_n} \phi(x) \text{tr}(X'_{s_1} \dots X'_{s_n} F_{mn}^2)$ [81,82]. As was pointed out in [83], to get SYM operators from short multiplets of $\mathcal{N} = 4$ superconformal algebra one should symmetrize products of X ’s (or of their $\mathcal{N} = 1$ superfield counterparts). This prescription is also supported and made more universal by the analysis of the supersymmetric versions of the above higher momentum dilaton operators in [84]. Note also that powers of X and F are related by linearized $\mathcal{N} = 4$ supersymmetry (in particular, $\text{Str}(XXXX)$ is related to $\text{Str}(FFFF)$ [85,86]). This suggests that in the general external background case the symmetrized trace in the NBI action should apply to both the components of 10-d field strength F_{mn} , $D_m X_s$, $[X_r, X_s]$ and powers of X_s in the Taylor expansion of the background fields.

³⁵ Some quantum corrections in such non-abelian D-brane actions were discussed in [80].

7. Derivative corrections to Born-Infeld action in open superstring theory

The field strength derivative corrections to the leading Born-Infeld part of the open string effective action can be computed using either S-matrix or sigma model (path integral or beta-function) approach. The advantage of the latter is that though it is based on the expansion in derivatives $\partial_k F_{mn}$, in the abelian case it is still non-perturbative in powers of F_{mn} . As explained in [19,20], one may start with the path integral expression (2.1) or (6.1),(6.3), expand the boundary action in powers of ∂F and compute the resulting correlators with the F_{mn} -dependent (super)propagator on the disc which in the superstring case has the form [21,19,20] (cf. (6.2))

$$\begin{aligned}\hat{G}_{mn} &= G_{mn}(\varphi_1, \varphi_2|F) - \theta_1 \theta_2 K_{mn}(\varphi_1, \varphi_2|F) \\ &= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{e^{-\epsilon n}}{n} (f_{mn} \cos n\varphi_{12} - i h_{mn} \sin n\varphi_{12}) \\ &\quad - \frac{i}{\pi} \theta_1 \theta_2 \sum_{r=1/2}^{\infty} e^{-\epsilon r} (h_{mn} \cos r\varphi_{12} - i f_{mn} \sin r\varphi_{12}) ,\end{aligned}\tag{7.1}$$

$$f_{mn} \equiv [(\eta - F^2)^{-1}]_{mn} , \quad h_{mn} \equiv F_{mk} f_{kn} = [F(\eta - F^2)^{-1}]_{mn} ,$$

where we have absorbed $2\pi\alpha'$ into the matrix F_{mn} and $(\eta - F^2)_{mn} \equiv \eta_{mn} - F_{mk}\eta^{kl}F_{ln}$. After the renormalization of the partition function $Z(A)$, i.e. eliminating logarithmic divergences by a redefinition of the vector potential,³⁶ one finds [20] that the resulting effective superstring Lagrangian has the following structure (we again suppress the factors of $2\pi\alpha'$ which multiply each F_{mn} and each pair of derivatives)

$$L = \sqrt{-\det(\eta_{mn} + F_{mn})} \left[1 + \mathcal{F}^{klmnpqcd}(F) \partial_k \partial_l F_{mn} \partial_a \partial_b F_{cd} + O(\partial^6) \right] ,\tag{7.2}$$

where the function $\mathcal{F}(F) \sim F^2 + F^4 + \dots$ can be, in principle, computed exactly. The leading F^2 term in \mathcal{F} is easy to find by comparing this action with the momentum expansion of the standard expression [5] for the 4-vector superstring amplitude on the disc [20]

$$L = \sqrt{-\det(\delta_{mn} + F_{mn})}$$

³⁶ This corresponds to subtraction of massless exchanges in the string amplitudes. Renormalization scheme ambiguity corresponds to field redefinition ambiguity in the S-matrix approach [19].

$$\begin{aligned}
& -\frac{1}{96} \left(\partial_a \partial_b F_{mn} \partial_a \partial_b F_{nl} F_{lr} F_{rm} + \frac{1}{2} \partial_a \partial_b F_{mn} F_{nl} \partial_a \partial_b F_{lr} F_{rm} \right. \\
& \left. - \frac{1}{4} \partial_a \partial_b F_{mn} F_{mn} \partial_a \partial_b F_{lr} F_{lr} - \frac{1}{8} \partial_a \partial_b F_{mn} \partial_a \partial_b F_{mn} F_{lr} F_{lr} \right) + O(\partial^4 F^6) . \quad (7.3)
\end{aligned}$$

The $FF\partial\partial F\partial\partial F$ term in (7.3) defines a super-invariant which appears also in other contexts (e.g., as a divergent 1-loop correction to quantized supersymmetric Born-Infeld action [48,49]). Starting with (7.3) in 10 dimensions and applying dimensional reduction as in (2.3),(6.15) one may determine the corresponding higher-derivative string α' corrections to D-brane actions.³⁷

For comparison, in the bosonic string case one finds [19,20]

$$\begin{aligned}
L &= \sqrt{-\det(\eta_{mn} + F_{mn})} \left[1 + \mathcal{F}^{kmnacd}(F) \partial_k F_{mn} \partial_a F_{cd} + O(\partial^4 F^k) \right] \\
&= \sqrt{-\det(\eta_{mn} + F_{mn})} \\
&- \frac{1}{48\pi} \left(F_{kl} F_{kl} \partial_a F_{mn} \partial_a F_{mn} + 8 F_{kl} F_{lm} \partial_a F_{mn} \partial_a F_{nk} - 4 F_{la} F_{lb} \partial_a F_{mn} \partial_b F_{mn} \right) + O(\partial^2 F^6), \quad (7.4)
\end{aligned}$$

where the leading F^2 term in the function $\mathcal{F}(F)$ can be found from the string 4-point amplitude [20] or from the 2-loop beta-function computation [88].

8. Acknowledgements

I am grateful to I. Buchbinder, F. Gonzalez-Rey, I. Klebanov, H. Liu and R. Metsaev for useful discussions of related questions. This work was supported in part by the PPARC, the EC TMR programme ERBFMRX-CT96-0045, INTAS grant No.96-538, and NATO Collaborative Linkage Grant PST.CLG 974965.

Appendix A. Born-Infeld action from string partition function on a disc

Let us review the original computation [6] of the partition function (2.1) in the abelian $F_{mn} = \text{const}$ background in a slightly generalized form: we shall assume that the boundary

³⁷ Note that the $O(\partial^2 X)$ higher-derivative terms discussed in [87] can be redefined away [50]: the scalar terms in [87] are related by field redefinition and integration by parts to $(\partial X)^4/X^4$ terms which accompany F^4/X^4 terms (cf. (4.28)) in the $\mathcal{N} = 4$ SYM 1-loop effective action.

part of the string action contains also the usual ‘particle’ term $(\dot{x}^m)^2$. The full (Euclidean) string action on the disc is then

$$I = \int d^2\sigma \frac{1}{2} T \partial^a x^m \partial_a x^m + \int_0^{2\pi} d\varphi \left[\frac{1}{2} M_{mn} \dot{x}^m \dot{x}^n - i \dot{x}^m A_m(x) \right]. \quad (\text{A.1})$$

Here M_{mn} may be interpreted as a condensate of the open string massive mode; taken at an ‘off-shell’ value $M_{mn} = \text{const}$ this term breaks conformal invariance of the sigma model. In what follows we shall set $M_{mn} = M \delta_{mn}$ and treat this term as a formal ‘regularization’ of the boundary kinetic operator. Integrating over the values of the string coordinate at internal points of the disc we arrive at the following effective action at the boundary of the disc (we isolate the constant zero mode $x_0^m = x^m - \xi^m$)

$$I_{\text{bdry}} = \frac{1}{2} \int_0^{2\pi} d\varphi \left[T \xi^m G^{-1} \xi^m + M \dot{\xi}^m \dot{\xi}^m + i F_{mn} \xi^n \dot{\xi}^m \right], \quad (\text{A.2})$$

where $\xi^m = \sum_{n=1}^{\infty} (a^m \cos n\varphi + b^m \sin n\varphi)$ and the scale-invariant (‘first order’) non-local operator G^{-1} is the inverse of the restriction of the Green function on the disc to its boundary,

$$G(\varphi_1, \varphi_2) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \cos n\varphi_{12}, \quad G^{-1}(\varphi_1, \varphi_2) = \frac{1}{\pi} \sum_{n=1}^{\infty} n \cos n\varphi_{12}. \quad (\text{A.3})$$

The action (A.2) thus contains both effectively first-order ($\sim T$) and second-order ($\sim M$) derivative parts and interpolates between the string-theory case $T \neq 0, M = 0$ which was discussed in [6] and the standard particle case $T = 0, M \neq 0$ which appeared in the Schwinger computation of $\log \det(-D^2(A))$. The resulting partition function will interpolate between the Born-Infeld (“ $\sqrt{1 + F^2}$ ”) and Schwinger (“ $\frac{F}{\sinh F}$ ”) results.

Putting F_{mn} in the block-diagonal form and concentrating on the first (1, 2) block we find, integrating over the coordinates ξ^1, ξ^2 as in [6] ($F_{12} = F$):

$$Z_{12} = Z_{12}(M) Z_{12}(F, M),$$

$$Z_{12}(M) \sim \prod_{n=1}^{\infty} (Tn + Mn^2)^{-2} \sim M \left[\prod_{n=1}^{\infty} \left(1 + \frac{TM^{-1}}{n} \right) \right]^{-2}, \quad (\text{A.4})$$

$$Z_{12}(F, M) = \prod_{n=1}^{\infty} \left[1 + \frac{F^2}{(T + Mn)^2} \right]^{-1}. \quad (\text{A.5})$$

$Z_{12}(F, M)$ depends only on the ratios $T^{-1}F$ and $T^{-1}M$. We shall ignore the (divergent) F -independent factor $Z_{12}(M)$ which can be absorbed into the renormalization of tachyon coupling at the boundary. When $T = 0$ the factors in the product in $Z_{12}(F, M)$ are n -independent, and using the regularization prescription $\prod_{n=1}^{\infty} c = c^{-1/2}$ as in [6] (with linear divergence again absorbed into the tachyon coupling [19]) we get the Born-Infeld result $Z_{12}(M = 0, F) = \sqrt{1 + (T^{-1}F)^2}$. When $T = 0$ we get the Schwinger result $Z_{12} = \frac{\pi M^{-1}F}{\sinh(\pi M^{-1}F)}$.³⁸ In general, for $M \neq 0$, $Z_{12}(F, M)$ is ‘more convergent’ than for $M = 0$ (i.e. M plays the role of an effective regularization parameter) and is given by a combination of Γ functions

$$Z_{12}(F, M) = \frac{\Gamma(\frac{T+M+iF}{M}) \Gamma(\frac{T+M-iF}{M})}{[\Gamma(\frac{T+M}{M})]^2} . \quad (\text{A.6})$$

In the case of the electric field background ($iF \rightarrow E$) the partition function (A.6) becomes

$$Z_{12}(E, M) = \frac{\Gamma(\frac{T+M+E}{M}) \Gamma(\frac{T+M-E}{M})}{[\Gamma(\frac{T+M}{M})]^2} . \quad (\text{A.7})$$

This partition function is well-defined for $|E| < T + M$, which is a generalization of the critical field strength condition ($|E| < T$) for the Born-Infeld action.

The general expression for the partition function is thus given by the product of factors for each eigenvalue F_p of the field strength

$$Z(F, M) = \prod_{p=1}^{D/2} \left[\frac{\Gamma(\frac{T+M+iF_p}{M}) \Gamma(\frac{T+M-iF_p}{M})}{[\Gamma(\frac{T+M}{M})]^2} \right] . \quad (\text{A.8})$$

It is easy to see that indeed

$$Z(F, M)|_{T \rightarrow 0} \rightarrow \prod_{p=1}^{D/2} \frac{\pi M^{-1}F_p}{\sinh(\pi M^{-1}F_p)} . \quad (\text{A.9})$$

At the same time, taking here the limit $M \rightarrow 0$ and using the Stirling formula $\Gamma(z \rightarrow \infty) = \sqrt{\frac{2\pi}{z}} (\frac{z}{e})^z [1 + O(\frac{1}{z})]$ we find that

$$Z(F, M \rightarrow 0) = \prod_{p=1}^{D/2} \sqrt{1 + (T^{-1}F_p)^2} [1 + (T^{-1}F_p)^2]^{\frac{T}{M}} \left(\frac{1 + iT^{-1}F_p}{1 - iT^{-1}F_p} \right)^{\frac{iF_p}{M}} [1 + O(M)] . \quad (\text{A.10})$$

³⁸ In the 1-loop field theory computation context $M = \pi/s^2$ where s is the proper time parameter which one is still to integrate over.

M thus plays here the role of a cutoff. Eq. (A.10) reduces to the Born-Infeld expression after renormalizing the divergent term in the exponent

$$Z(F, M \rightarrow 0) = \sqrt{\det(\delta_{mn} + T^{-1}F_{mn})} e^{\frac{1}{M}f(F)}. \quad (\text{A.11})$$

The linearly divergent term cancels out in the superstring case as in (6.7).

The superstring generalization of the action (A.2) contains three extra fermionic terms:

$$\frac{1}{2} \int d\varphi (T\psi^m K^{-1}\psi^m + M\psi^m \dot{\psi}^m + iF_{mn}\psi^m \psi^n), \quad (\text{A.12})$$

where K is defined in (6.2). As a result, (A.5) is replaced by

$$Z_{12}(F, M) = \frac{\prod_{r=1/2}^{\infty} [1 + \frac{F^2}{(T+Mr)^2}]}{\prod_{n=1}^{\infty} [1 + \frac{F^2}{(T+Mn)^2}]} = \prod_{n=1}^{\infty} \left[\frac{1 + \frac{F^2}{(T-\frac{1}{2}M+Mn)^2}}{1 + \frac{F^2}{(T+Mn)^2}} \right]. \quad (\text{A.13})$$

Then the generalization of (A.8) becomes ($D = 10$)

$$Z(F, M) = \prod_{p=1}^{D/2} \left[\frac{\Gamma(1 + \frac{T+iF_p}{M})}{\Gamma(\frac{1}{2} + \frac{T+iF_p}{M})} \frac{\Gamma(1 + \frac{T-iF_p}{M})}{\Gamma(\frac{1}{2} + \frac{T-iF_p}{M})} \frac{[\Gamma(\frac{1}{2} + \frac{T}{M})]^2}{[\Gamma(1 + \frac{T}{M})]^2} \right]. \quad (\text{A.14})$$

This expression is regular in the $M \rightarrow 0$ limit and reduces simply to the Born-Infeld action³⁹

$$Z(F, M \rightarrow 0) = \sqrt{\det(\delta_{mn} + T^{-1}F_{mn})} [1 + O(M)]. \quad (\text{A.15})$$

Thus M plays indeed the role of a natural regularization parameter. In the limit $T \rightarrow 0$ one finds $Z(F, M) = \prod_{p=1}^{D/2} \frac{\pi M^{-1}F_p}{\tanh(\pi M^{-1}F_p)}$.

Similar expression for the partition function can be found using the Green-Schwarz light-cone gauge string action as in [17]. Assuming that the vector field strength has only spatial (magnetic) components F_{ij} one is to replace (A.12) by $\frac{1}{2} \int d\varphi (TS^a K^{-1}S^a + MS^a \dot{S}^a + i\hat{F}_{ab}S^a S^b)$, where S^a is an $SO(8)$ spinor, $\hat{F}_{ab} = \frac{1}{4}(\gamma^{ij})_{ab}F_{ij}$ and K is again given by (6.2) (S^a is the restriction of the l.c. GS spinor variable which is a 2-d spinor to the boundary of the disc and thus like ψ^m is antiperiodic in φ [16]). The resulting partition function has the same form as (A.14) (with $D = 8$, i.e. $p = 1, 2, 3, 4$) but with F_p in the fermionic contributions (two Γ -function factors in the denominator) replaced by the eigen-values \hat{F}_p of the matrix \hat{F}_{ab} , $\hat{F}_1 = \frac{1}{2}(-F_1 + F_2 + F_3 + F_4)$, etc. (see [89] for a

³⁹ The fermionic factors give only divergent contributions $([\Gamma(\frac{1}{2} + z)]_{z \rightarrow \infty} \rightarrow \sqrt{\frac{2\pi}{z}} (\frac{z}{e})^{\frac{1}{2}+z} [1 + O(\frac{1}{z})] \rightarrow z^z)$ that cancel the $\frac{1}{M}$ divergences coming from the bosonic sector.

discussion of a similar partition function on the annulus). The $M \rightarrow 0$ limit of the resulting partition function has again the Born-Infeld Lagrangian as its finite factor, but as in the bosonic case (A.11) and in contrast to the NSR case (A.15) the divergent $\frac{1}{M}$ terms here do not cancel, i.e. in this case the M -regulator does not seem to preserve the world-volume supersymmetry (the same conclusion is reached in the case of the exponential regulator used in (6.2)). At the same time, the formal ζ -function regularization implies [6,16] that the bosonic factor is the BI one while the fermionic contribution is simply equal to 1 ($\zeta(0, 1) = \zeta(0) = -\frac{1}{2}$, $\zeta(0, \frac{1}{2}) = 0$).

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